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Mathematics and Literature (the sequel)

Imagination as a pathway to advanced mathematical ideas and philosophy

Introduction

This article is the sequel to the use of *Flatland* with beginning algebra students reported in Sriraman (2003). The use of *Flatland* with beginning algebra students resulted in the positive outcomes of cultivating critical thinking in the students as well as providing the teacher with the context necessary to introduce sophisticated mathematical ideas. The marriage of mathematics and literature led students to reflect on contemporary society and its problems as well as gain an insight into notions of limits, historical approximation techniques and various non-Euclidean geometries (fractal geometry and Minkowskian space-time geometry). This atypical but refreshing learning experience led students to request the reading of one of the sequels to *Flatland*. The 'providential' release of Stewart's *Flatterland* in 2001 seemed like the ideal follow up to *Flatland*. Banchoff's (2001) review of *Flatterland* for the Mathematical Association of America partially found in the back cover states: '*Flatterland* challenges readers to go beyond *Flatland* and deal with phenomena, not just in dimensions higher than four, but in many exotic geometric realms that stretch our imagination and powers of visualization.' Upon reading the book over the winter break, my personal impression was that the ideas introduced in the entire book would pose a challenge to university math and physics majors. However the material in the first five of the eighteen chapters were within the scope of 13–14 year-old ninth graders. In fact, some of the ideas introduced in the first five chapters such as arbitrary dimensions in mathematics, and fractal geometry had been discussed in the class during the reading of *Flatland*. In addition, Stewart (2001) had brilliantly made modern ideas such as encryption on the Internet, the taxi-cab metric, and fractal dimensions, among others, very accessible to the lay person. This was achieved by creating a contemporary setting in which the heroine Vikki, the great-great-granddaughter of *Flatland*'s protagonist A. Square, is taken on a guided virtual reality tour of the mathematical universe by a space hopper.

Setting up the experiment

I was fortunate once again to have the support of the principal in this endeavour and was supplied with a classroom set of *Flatterland*. I decided to make use of the book towards the end of the high school year, when interest in the regular curriculum typically begins to wane. The pedagogical reason for doing this was to ensure that students had the mathematical background (from the algebra curriculum) necessary to allow me to develop the ideas in the book. My goal was to use *Flatterland* as scaffolding in order to:

1. explore non-intuitive math problems;
2. further students' understanding of dimension;
3. to help students gain a deeper under-

standing of fractal geometry; and
4. to develop the taxi-cab geometry.

The first five chapters of *Flatterland* were read sequentially in April–May 2002. The reading structure was very similar to that used in the reading of *Flatland*. As in the experiment with *Flatland* (Sriraman, 2003), reading, writing and discussing the book replaced one of the tests in the second semester, which again went over very well with the students.

I explained that the book was a recently released fun sequel to *Flatland*. The five chapters were read over eight weeks. A reading schedule was provided for the students. We discussed and developed the ideas from the first five chapters in eight 50-minute Friday class periods. Unlike the previous experiment where students were split into groups for discussion, I used a show of hands to separate the class into two groups, namely those that 'thought' they understood the reading and those that were confused. Then I applied the Socratic method of question-hypothesis-elenchus-acceptance/rejection to moderate the classroom discussion. In other words classroom discussion began with a 'confused' student stating the nature of their confusion, which was then restated as question. Then unconfused students were asked to respond to the question. Their response/explanation was used to generate a testable hypothesis, which was subject to elenchus (or refutation), eventually leading to acceptance of the explanation or rejection, in which case the hypothesis was re-examined. This process was modelled many times in the regular algebra curriculum in order to set the stage for the discussion of *Flatterland*. The fascinating outcomes of the use of *Flatterland* with the 13–14 year old ninth graders is presented in the next sections.

The times they are a-changin'[1]

The first three discussions revolved around the contemporary setting of *Flatterland*, the dashing heroine Vikki, and strange packing problems. Having read *Flatland* in the first semester, students were able to use to the contemporary setting of the book and compare the changes described in the book to the tidal waves of change in the 20th century. First, the girls in the class were pleased that the protagonist was a female, who was roughly their age. Second, the students were happy to see that the chauvinistic society of *Flatland* had evolved and embraced women's liberation due to various Flatlandian wars and revolutions. Third, students really enjoyed the satirical word play of the book and relished the double-edged nature of words used by Stewart (2001). Student comments that made various critical comparisons on the changes in *Flatland* and their parallels to our world are summarised below.

> [In *Flatterland*] Flatland has changed in quite a few ways. They have become more intelligent and not so prejudiced against women. Women are treated as equals rather than just unintelligent things. They send messages like us (e-mail) and code stuff… I'd say *Flatterland* is the more mature version of *Flatland*.
>
> *Flatterland* now has interline computers like our Internet computers, telephones and many other devices that were actually invented in the past century. Women have gained more respect and rights in *Flatterland* just like the voting movements (suffrage) in our history… but they still have to sway their back from side to side and sing to avoid hurting other people.
>
> It still seems as though people won't accept the fact of there being a 3rd dimension. You are deemed 'weird' if you have a radical idea, and it still happens today in our society when people don't accept new ideas from science.
>
> Did you notice that generations now do not gain another edge with each generation. I like the word play, like you don't have an edge over others just cause you're born into a rich family. But we all know in reality you do have an edge if your parents are rich.

1 The title of a revolutionary song by Bob Dylan, released in January 1964 during the Civil Rights movement in the US.

Is everything mathematically possible? Packing fruit and dementia with dimensions

The third chapter of *Flatterland* describes the visitation of the 'Space hopper' and Vicki's discovery of the space hopper's strange shape. The use of a time-series of cross-sections was understood by the students and thought similar to the discovery of the Circular visitor's shape by A. Square. Just as a sphere moving through a horizontal plane would first be seen as a dot followed by a series of expanding circles and then by a series of contracting circles culminating in a dot and then disappearance; students were able to extend this notion and accept that it was plausible to think of a series of expanding and contracting moving spheres as the shadow a 4-dimensional hypersphere would cast in our world. Students found some of the ideas in the third and fourth chapters a little difficult to understand. For instance, Stewart (2001) describes the non-intuitive possibility of fitting a cube of side length 1.06 into a unit cube![2] One of the students actually tried to accomplish this by using thin cardboard boxes but was unsuccessful. This was discussed in the class and students concluded that it was practically 'impossible' to find instruments that could be used to make a cube of side length 1.06. Therefore they deemed this non-intuitive notion of squeezing a larger cube into a smaller one as 'mathematically' possible but practically 'impossible'. I used this opportunity to discuss the need for calibration and accurate measuring instruments in science to generate data that could verify or refute hypothesis. Many students were also unable to completely understand the encryption procedure described by Stewart (2001) to encode and decode messages. However this presented the opportunity to introduce the binary numeral system and play with the four basic arithmetic operations in base 2. Students really enjoyed this and realised the arbitrariness of numerals.

Students were also intrigued and understood the fruit packing problem described by Stewart (2001), namely what is the most efficient way to pack fruit (that are roughly spheres) into cardboard boxes that are rectangular prisms? This problem allowed us to explore approximation techniques to determine the maximum amount of a fruit that could be packed into a box, given particular starting assumptions of the size of fruit, and the size of boxes. We choose starting sizes for apples, grapefruit, pumpkins and watermelons to determine how many could be packed in a standard fruit box found in the grocery store. One student took the initiative of going to a grocery story and bringing some perforated sheets used in apple boxes, which visually demonstrated the practical nature of the packing problem. Student comments follow:

> It seems that it is mathematically possible to do anything. I don't believe you can fit a bigger cube into a smaller one… But I really like the idea of using binary digits for changing letters to symbols with only 0 and 1 and it made sense how to spot the errors.

> In Spaceland or 'Planeturthian' you have more space than in Flatland. They are forced to stick oranges with gaps in Flatland… but we can stack them differently and reduce the gaps because we can move in more directions than they can.

> It makes sense to add a dimension every time you can move something in a different direction. So we can have a 'chalk-cheese' direction just like north-south, east-west, up-down. The idea of finding the dimension also makes sense because if you had a 3D ball, its surface is 2D, because $3 - 1 = 2$. So the surface is always one dimension smaller than the original dimension. So a 101 dimensional ball has a 100D surface.

> The idea of stacking spheres to make a hypersphere is really far out; but it makes sense if you think of making a sphere by stacking together smaller and bigger circles

2 Jerrard & Wetzel (2004) recently wrote an expository article on the history of this problem. They wrote that the origin of this problem was a wager made and won by Prince Rupert in the late 17th century that a hole could be made in cube such that another cube of the same size could slide through. One hundred years later Nieuwland calculated the dimensions of a larger cube that could pass through a cube of unit side by determining the size of the largest square that fits in this cube.

Fractals, taxicabs and square circles

The fourth chapter of *Flatterland* describes the arbitrary nature of dimensions in mathematics by surveying ideas from different geometries. In *Flatland*, the notion of self-similarity was used by the circular visitor from Spaceland to postulate the existence of higher dimensions to A. Square. This notion was understood by students as was evidenced in their constructions of 4-dimensional objects such as a hyper-cube and in their argument that the geometric sequence 1, 2, 4, 8, 16… that counted the number of vertices of self-similar objects indicated the existence of higher dimensional objects (even if one could not visualise them). During the previous experiment with *Flatland*, I had used students' notion of self-similarity to construct the Sierpinski triangle and posed the question about the dimension of fractal objects. In *Flatterland*, the notion of fractal dimension is explored at a deeper level. One of the classroom discussions revolved around the calculation of fractal dimensions. Table 1 summarises student strategy to calculate fractal dimensions.

Students calculated the dimensions of the Sierpinski triangle and the Koch snowflake using trial and error on their calculators. Since they were not exposed to the notion of logarithms, I decided that it was pedagogically sound to perform this calculation by trial and error. The dimensions found for the Sierpinski triangle and Koch snowflake were 1.584 and 1.261 respectively, which was accepted by the class as being accurate. One industrious student located a fractal website, which we used to check our dimension calculations.

In the consequent discussion of the Mandelbrot set, Stewart (2001) makes use of the taxicab metaphor to describe coordinates in the complex plane. The description given to Vikki about the moves necessary to get from one point to another in Quadratic City created the perfect setting to talk about taxicab geometry. Since students had been exposed to the

Table 1. Extension of self-similarity idea to discovery of formula for calculating fractal dimensions.

Object	Dimension of Object	Self-similar copies made	Pattern
Point	0	1	$1 = 2^0$
Line segment [2 points]	1	2	$2 = 2^1$
Square [two segments] [4 points]	2	4	$4 = 2^2$
Cube [Two squares] [8 points]	3	8	$8 = 2^3$
Hypercube [Two cubes] [16 points]	4	16	$16 = 2^4$
[Student observation: The dimension always shows up in the exponent]			
Sierpinski Triangle	don't know	3	$3 = 2^{\text{dimension?}}$
Sierpinski Triangle	Know (1.584)	3	$3 = 2^{1.584}$
Koch Snowflake	don't know	4	$4 = 2^{\text{dimension?}}$
[Student observation: The dimension of the Koch Snowflake cannot be the same as a square. The size of the snowflake does not double, but triples every time]			
Koch Snowflake	Know (1.261)	4	$4 = 3^{\text{dimension}}$ $4 = 3^{1.261}$

distance formula in the analytic geometry segment of the algebra curriculum, it was quite easy to introduce the taxi-cab metric by analogy.

We explored several questions related to *Flatland* and middle school geometry, namely are there any regular polygons in taxicab geometry? In particular we compared the notions of betweenness in the Euclidean and taxicab metrics and explored various Flatland shapes such as isosceles triangles, squares, and circles in the taxicab metric. A vignette of this exploration follows.

A classroom vignette

Teacher: How do we determine whether one point is between two other points?

Student 1: You plot the points and see where they are on the real number line.

Student 2: Couldn't we use the midpoint formula?

Student1: But you can't be sure that this point is exactly in the middle of the two other points.

Teacher: Don't we need some information about the location of the points to use the midpoint formula?

Student 3: How can you be sure that the points are on the real line? Can't the points be outside the real line?

Student 2: Yeah, we use (x,y) co-ordinates to locate the points.

Teacher: Why don't we look at an example. What if we take the points P(3,2), Q(6,4) and R(9,6) and plot them?

Student 4: The points are on the line with slope 2/3.

Teacher: Okay, now how do we check whether Q is between P and R?

Student 1: Just use the plot and you see the point Q is right in between.

Teacher: What if you don't have graph paper and you can't plot the points?

Student 2: You can just visualise it in your head.

Teacher: Can we use any formulas we learned?

Student 4: The midpoint formula?

Teacher: But you can't always be sure that one point will always be between the other two.

Student 5: Why don't we look at distances between the three points?

Teacher: That's a good idea. Does anybody remember how we calculate distances?

Student 6: The distance formula.

Teacher: Yes, but how can we use the distance formula to decide that Q is between P and R?

Student 6: Calculate PQ, then QR, and then see if they add up to PR.

Teacher: Does anybody disagree with this idea?

Student 4: Yeah, but does it always work?

Student 6: I think it does. You can check it on the real line if you like.

Teacher: If it works on the real line, do you think it works on every line?

Student 6: Yeah, cause the real line is just another line with no slope.

[Students perform calculations and determine that $PQ + QR = PR$.]

Teacher: Can we define betweenness now?

Student 6: We already did. Just calculate the three distances and see if the two smaller ones add up to the total distance.

Teacher: Okay, so we say that a point B is between A and C if $AB + BC = AC$, and to make life easier we write A-B-C. Now the question is does it work the same way in the taxicab world?

Student 5: But how are we going to calculate distances there? Don't we need that?

Student 7: You can just count on a graph paper.

Teacher: Can we come up with a formula maybe? Just like the distance formula we already know?

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

[Classroom discussion eventually leads to the discovery that distance in the taxicab world is calculated by counting the number of blocks travelled either east-west plus the number of blocks travelled up-down.]

Teacher: We can write this formula as $d_T = |(x_1 - x_2)| + |(y_1 - y_2)|$ and

we'll use d_E for the normal way of calculating distance. Now can we check whether Q is between P and R?

Student 8: Do we do the same thing like before?

Teacher: What do you mean?

Student 8: Like see if those three distances add up?

Teacher: What does the class think?

Student 6: I think it will be the same.

Teacher: Same what?

Student 6: Like the same rule, you know $AB + BC$ has to equal AC.

[Calculations reveal that betweenness does work out the same way.]

Student 7: Can't we go from P to R like in a city, where you are trying to avoid a block. What I'm saying is can't you go through a different point, like some point X and get to R. Does that mean there are other points P and R that are between but not on the line?

[This elenchus (refutation) led to a discussion of the difference between 'metric' betweenness and betweenness as defined in Euclidean geometry. The discussion led us to re-examine the Euclidean hypothesis for betweenness and reach the following conclusion.]

Teacher: We will impose the requirement in the definition of betweenness that points be on a particular line to take care of this problem of 'metric' betweenness in taxicab geometry. This way we can use the same definition in both geometries

Commentary on vignette

This vignette is used for the purpose of illustrating the Socratic method. As is evident in the transcribed comments, students were unwilling to accept statements made by other students and the teacher blindly but subjected it to scrutiny. As teachers we should value the pedagogical value of the Socratic method even though it can be very tedious on occasions. The preceding vignette is a condensed and edited version of 45 minutes of dialogue eventually leading to the acceptance of the taxicab metric and the notion of betweenness. In the discussion in the following week we used the taxicab metric to explore regular polygons and circles.

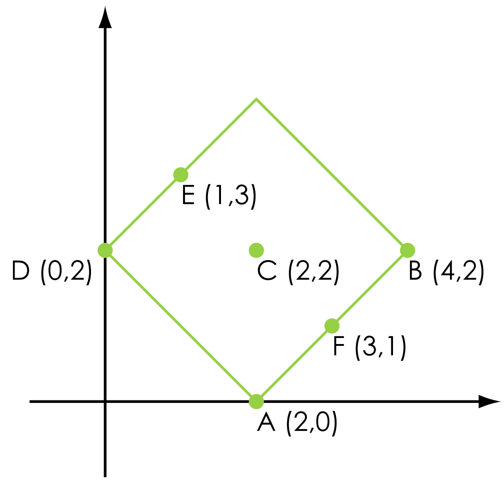

Figure 1

Students were 'blown away' by the bizarre appearances of known Euclidean objects in the taxicab metric. In addition to discovering circles appeared as squares in the taxicab metric, the class also found several known triangles to have strange properties; for example, equilateral triangles in the Euclidean metric turned out to be isosceles but equiangular in the taxicab metric!

A foray into philosophy

Towards the end of the fifth chapter, one of the students in the class asked the ponderable question, 'Is mathematics real?'. The fact that Vikki was zipping in and out of different geometrical worlds in mathiverse (the mathematical universe) led many students to wonder whether mathematics was invented or discovered. In other words, did Vikki discover geometries that were present *a priori* or were the different geometries a figment of the space hopper's imagination made real via the use of virtual reality? These questions were raised by several students who were big fans of the movie *The Matrix* in which reality as is was quite different from reality experienced through a virtual interface. These questions posed the challenge of introducing mathemat-

ical philosophy at an elementary level. The following vignettes illustrate 13–14 year-old student viewpoints on the nature of mathematics. I have condensed two students' expressions of their viewpoints, which parallel the Platonist and Formalist viewpoints of the nature of mathematics. It is interesting to note that the class was evenly divided between the Platonist and Formalist camps as in professional mathematics, which I believe was a function of the non-intuitive mathematics brought alive by the book.

> Mathematics is something real. I strongly feel this way because you can never prove math wrong. Every equation in mathematics has some connection to the real world. Let's use area of a square or a rectangle for example, you can always use this in the real world… and I'm sure that there is a mathematical equation that solves the packing orange problem. So for the various reasons I have stated I believe the mathematical world to be real.

[This quote shows similarities to the Platonist view of mathematics.]

> I believe mathematics is imagined. For one, nothing on earth is perfectly predictable… like take the weather. I don't think there is any mathematical equation that can predict the weather. Mathematics is simply there to entertain people… it sometimes helps explain things like how to calculate area without measuring, and there are laws like Newton's laws of motion that solve some problems but not all. Math is food for thought… it can interest some people and easily scare others away.

[This quote shows similarities to the Formalist view of mathematics.]

Conclusion

The experiment with *Flatterland* lasted eight weeks towards the end of the school year. Just when student interest in the regular algebra curriculum had begun to wane, the book served as a catalyst to renew their enthusiasm for mathematics. The anticipation of being able to talk about the book on Friday kept student interest in the regular algebra curriculum alive. The ideas in the book deepened many students' interest in mathematics. It led three students to deciding to attend a summer math camp at a nearby university. Four of the students in the class are strongly considering a career that is related to mathematics or computer science. It is professionally satisfying to realise that the use of mathematics literature with students can lead to vistas unexplored and unimagined both for the students and the teacher.

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